

Dense coding with mixed state and steerability

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Abstract

Mixed state can be used in dense coding. We have analyzed here that maximally entangled mixed states like Werner state is dense codeable for a certain range of state parameter whereas for some wider range of the state parameter the state is 'steerable' but cannot be used in dense coding. For qutrit system we consider isotropic states and have found similar characteristics, like the states are 'steerable' for a range of state parameter but for a sub range of that steerability range only, the states are dense codeable.

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Coding is the process of transforming information during a communication process. The sender encodes the message and transmits the encoded information over a classical communication channel while the recipient decodes the encoded message. In quantum information science, an enthralling idea was generated first in [1] by showing the advantage in exchanging quantum bits ('qubits') through a quantum communication channel over sending classical bits ('0' and '1') via a classical communication channel. The idea of 'superdense coding' is based upon encoding two entangled qubits so as to enable sending four messages ('00', '01', '10', '11') and thereby sending 2 bits of information by manipulating just one of the qubits. The quantum channel shared by the sender and receiver may either be the triplet states ($|\Psi^+\rangle$ and $|\Phi^\pm\rangle$) or the singlet state ($|\Psi^-\rangle$), which are also called 'Bell-states'.

as

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} (|000\rangle_{ABC} + |111\rangle_{ABC}), \quad (2)$$

where, Alice (A), Bob (B) and Cliff(C) share the maximally entangled state (2). They however, showed that the success probability of such scheme is basically controlled by the measurement basis of the third particle as well as the entanglement and communication are also controlled in a quantum way. Hence the term 'controlled dense coding' was coined. We, recently have studied the applicability of this scheme on various other classifications of GHZ state in [3].

The idea of controlled dense coding can also be restricted to two parties as has been discussed in [4, 5]. In this dense coding, with the mixed basis for two qubits, which are defined as

$$\begin{aligned} |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \\ |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \end{aligned} \quad (1)$$

$$\begin{aligned} |\chi^a\rangle &= |\Psi^-\rangle \\ |\chi^b\rangle &= |\Psi^+\rangle \\ |\chi^c\rangle &= |00\rangle \\ |\chi^d\rangle &= |11\rangle, \end{aligned} \quad (3)$$

Hao *et. al* exhibited dense coding using a tripartite pure entangled state known as Greenberger-Horne-Zeilinger (GHZ) state in [2]. The GHZ state is defined

the sender Alice knows that some of her qubits are not coded as per her requirement, which she calls as 'wrong' qubits. She may either discard these or may

proceed with them for the success of the protocol. If she decides to proceed, then Alice has ‘control’ over her ‘wrong’ qubits in the sense that she knows what she sends. Then if Alice clones the ‘wrong’ qubits (photons) and sends it to Bob associated with a classical message on that while Bob corrects those ‘wrong’ message, the protocol can be termed also as ‘controlled superdense coding’. Recently a ‘deterministic mediated superdense coding with linear optics’ has been proposed in [6]. All of these above protocols have one common feature that the sender and the receiver must convince themselves that they share a maximally entangled state.

Let us now venture into another domain of information science, known as ‘Steering’. ‘Steerability’, a concept which is strictly stronger than non-separability and strictly weaker than Bell non-locality, [7, 8], was first brought to light by E. Schrodinger in [9], although the word Schrodinger used in place of ‘steering’ was actually ‘driving’ (or ‘controlling’). According to him *“a sophisticated experimenter can, by a suitable device which does not involve measuring non-commuting variables, produce a non-vanishing probability of driving the system into any state he chooses.”* A bit disturbing idea from the view point of classical physics, ‘steering’ tells one that, if two parties Alice (A) and Bob (B) share a pure entangled state, then one party (say) Alice can, by a making an appropriate measurement on her sub-system, create any pure quantum state $|\psi\rangle$ for Bob’s sub-system with probability $\langle\psi|\rho^{-1}|\psi\rangle^{-1}$, whenever that is well defined [9]. However, according to Einstein-Podolsky-Rosen (EPR) and Schrodinger, ‘steering’ may be demonstrated for any pure entangled state and this is also true of Bell non-locality. In other words one can say for pure states the concepts of ‘entanglement’, ‘steering’ and ‘Bell non-locality’ actually coincide [10].

In bipartite system, two parties, Alice and Bob, if share an entangled state $|\psi\rangle_{AB}$, then by measuring her subsystem, Alice can remotely change (i.e. ‘steer’) the state of Bob’s subsystem in such a way that would be impossible if their systems were only classically correlated. The simplest example of ‘steering’ is given by the maximally entangled state of two qubits $\frac{|00\rangle_{AB} + |11\rangle_{AB}}{2}$. Alice can project Bob’s system into the basis $\{|\alpha\rangle, |\alpha\rangle^\perp\}$ by making a measurement of her sub-system in the conjugate basis $\{|\alpha^*\rangle, |\alpha^*\rangle^\perp\}$. It has also been proved that any pure entangled state of

arbitrary dimension d is ‘maximally steerable’[11].

In controlled dense coding, the three parties Alice, Bob and Cliff, when share a GHZ state (2), then with appropriate choice of basis and subsequently applying von-Neumann measurement to his part of qubit, Cliff can make Alice and Bob share a non-maximally entangled state (which depends upon the parameter of Cliff’s basis) by classically informing Alice about his measurement result. Consequently Alice, on applying the auxiliary qubit to the system and then applying an appropriate collective unitary operation, can transform the bipartite non-maximal shared state between her and Bob to a state, where on performing again a von - Neumann measurement she ultimately shares a maximally entangled state with Bob (which obviously depends on her measurement result) [2]. Following these, we can immediately claim that Alice could actually ‘steer’ her part of qubit to achieve the desired goal of dense coding. So, in case of pure entangled states, the dense coding has actually been ‘steered’ (or ‘controlled’). Since we know that the protocol of dense coding requires the sharing of maximally entangled state between the sender and the receiver, Alice could actually ‘steer’ her part of qubit to establish a maximally entangled channel with Bob.

A shared quantum state is said to be ‘dense codeable’ if it is useful for dense coding. In $d_A \otimes d_B$ system, (where d_A is the dimension of the system of Alice and d_B is the dimension of the system of Bob), the ‘capacity of dense coding’ [12] for a given shared state ρ_{AB} is defined as

$$\chi = \log_2 d_A + S(\rho_B) - S(\rho_{AB}), \quad (4)$$

where, $S(\rho) = -\text{Tr}(\rho \log_2(\rho))$ denotes the von-Neumann entropy.

Ideal dense coding protocol involves a usage of prior maximal entanglement to send two bits of classical information by the physical transfer of a single encoded qubit. But when the prior entanglement is not maximal and the corresponding initial state of the entangled pair of qubits used for the dense coding is mixed, then such a protocol is known as ‘mixed state dense coding’. Such a kind was first shown by Bose *et. al* in [13].

The Werner state [14], which is a convex combination of a pure maximally entangled state and a

maximally mixed state, can be regarded as a maximally entangled mixed state, since the entanglement of formation [15, 16] of the state cannot be increased by any unitary transformation. The motivation of studying such a state lies in the fact that, these states can be generated in laboratory from a universal source entanglement, as has been discussed in [17]. In $2 \otimes 2$ dimensional system, Werner state can be expressed as,

$$\rho_{\text{werner}} = p |\psi^-\rangle\langle\psi^-| + (1-p) \frac{I_2 \otimes I_2}{4}, \quad (5)$$

where, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$ is the singlet state and I_2 is the 2-dimensional identity operator, where $|H\rangle$ and $|V\rangle$ are respectively horizontal and vertical polarization states. It has been shown in [18], that the state ρ_{werner} of eq. (5) is effective in dense coding only for $p \geq 0.7476$. Also it has been shown in [11] ‘steerability’ of the state (5) is a monotonic function of the state parameter p i.e. when p decreases, ‘steerable weight’ of the state decreases monotonically. The Werner state is ‘maximally steerable’ when $p = 1$ and becomes ‘unsteerable’ exactly when $p = \frac{1}{\sqrt{3}} \approx 0.57735$. Nevertheless, it was shown in [11], ‘steerability’ can be demonstrated for Werner state again with $p < \frac{1}{\sqrt{3}}$. We can illustrate our observations as shown in Figure 1.

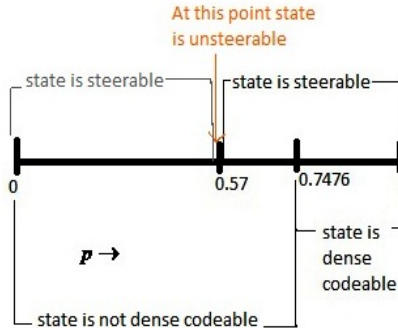


Figure 1: The above illustration shows that ρ_{werner} is unsteerable only for $p \approx 0.57$. For $0 < p < 0.57$ as well as for $0.57 < p \leq 1$, ρ_{werner} is steerable. But only for $p \in [0.7476, 1]$, the state is dense-codeable

From the above analysis we can claim that for $2 \otimes 2$ dimensional system, **dense-codeability of a mixed state does not depend on the state’s ‘steerability’**. It is easy to see that, $2 \otimes 2$ dimensional Werner state is both ‘steerable’ as well as dense-codeable for a

certain range of state parameter p whereas for some other range of p the Werner state is ‘steerable’ but not dense-codeable.

An isotropic[7, 19] state is a $d \otimes d$ dimensional two qubit quantum state that is invariant under any unitary transformation of the form $U \otimes U^*$, * denoting the complex conjugate. Werner state, however, is $2 \otimes 2$ dimensional case of an isotropic state. The isotropic state is a one parameter family of state and can be written as,

$$\rho_{\text{iso}} = p |\phi^+\rangle\langle\phi^+| + (1-p) \frac{I_d \otimes I_d}{d^2}, \quad (6)$$

where, $|\phi^+\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}} |i\rangle |i\rangle$. The state (6) is steerable if $p > \frac{H_d-1}{d-1}$ [7], where $H_d = \sum_{n=1}^d \frac{1}{n}$. We now consider the isotropic state in qutrit system, where we consider $d = 3$, which we denote by $\rho_{\text{iso}}^{\text{qutrit}}$. This state is ‘steerable’ if the parameter $p > 0.41665$.

From eq. (5) it is known that in $3 \otimes 3$ dimensional system a given state ρ_{AB} is dense codeable when capacity of dense coding of the state is more than $\log_2(3)$ or $S(\rho_B) > S(\rho_{AB})$, where $S(\rho_{AB})$ denotes the von-Neumann entropy of the state. It is easy to show that, for the state $\rho_{\text{iso}}^{\text{qutrit}}$, which will be of the form

$$\rho_{\text{iso}}^{\text{qutrit}} = p |\phi^+\rangle\langle\phi^+| + (1-p) \frac{I_3 \otimes I_3}{9}, \quad (7)$$

where $|\phi^+\rangle = \sum_{i=1}^3 \frac{1}{\sqrt{3}} |i\rangle |i\rangle$ is dense-codeable, since von-Neumann entropy of the subsystem of the state (7) is more than the von-Neumann entropy of the joint system but only when $p \geq 0.716$. As before the above analysis has been illustrated in Fig. 2.

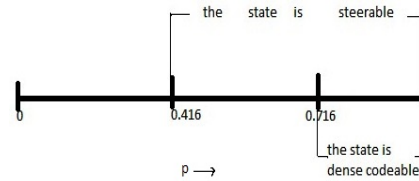


Figure 2: It is shown in the figure that the state (7) is dense codeable when $0.716 \leq p \leq 1$, but steerable only when $0.4166 \leq p \leq 1$, which shows in turn that steerability of the state does not imply the dense codeability.

In conclusion, we have examined the well known Werner state from two view points. One that of its ‘steerability’ and another is that of the state’s dense codeability. We have seen that, for a certain range the state is ‘steerable’, whereas the state is dense codeable for a subset of that range only. Again the qutrit form of the Isotropic states represented in (6) has been considered. Interestingly, it is observed that such a state is not dense codeable for a certain range of parameter which is however $p < 0.716$ but for $p \geq 0.716$ the state is dense codeable. Nevertheless the state is ‘steerable’ for a wider range of p , i.e. $0.416 < p \leq 1$, thus implying again that ‘steerability does not imply dense-codeability’ in qutrit system too.

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